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Open Quasi-Free Systems

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We compute the time evolution of an infinitely extended, open, quasi-free system in the weak coupling limit followed by the thermodynamic limit, and we derive the Onsager relations and the properties of entropy production.

KEY WORDS: Open system; quasi-free system; CCR algebra; CAR algebra; quasi-free state; completely positive map; contraction semigroup; boson reservoir; fermion reservoir; weak coupling limit; thermodynamic limit; dynamical semigroup; heat flow; matter flow; entropy flow; Onsager relations; entropy production.

1. INTRODUCTION

The aim of this work is the rigorous study of some microscopic models for the irreversible time evolution and the thermodynamic behavior of a large, open quantum system. Spohn and Lebowitz[1] have studied finite systems weakly coupled to several thermal reservoirs at different temperatures, and derived the Onsager relations for the heat flows, the positivity and convexity of the entropy production, and the principle of minimal entropy production in the regime of linear thermodynamics. For infinite systems, a similar study would be extremely difficult in general. Therefore, as usual, we turn for some insight to the quasi-free case. We consider first a spatially confined, quasi-free system and adapt Davies' techniques[2,3] to study its evolution in the limit of weak coupling to several reservoirs. Then we take the thermodynamic limit in the case in which the reduced evolution is translationally invariant. To obtain this, we are compelled to use couplings which are themselves translationally invariant, so that the system is in contact with all the reservoirs at each point in

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space in the same way. The time evolution of local observables and the stationary states are computed explicitly. We also discuss a model in which the coupling is not translationally invariant and the points of contact between system and reservoirs move to infinity in the thermodynamic limit. Then, in order to see the effect of the reservoirs on the evolution of the local observables, one must scale the time proportionally to the size of the system.

For a class of quasi-free states determined by functions of the one-particle Hamiltonian, we are able to compute explicitly the relevant thermodynamic quantities (heat and matter flows, entropy production) and their densities in the thermodynamic limit, and find that all results obtained by Spohn and Lebowitz in the linear approximation are also valid here. Moreover, as already remarked, we find the exact expression of the stationary nonequilibrium state, which is determined by the initial states of the reservoirs and by the relative strengths of the couplings. It is not a thermal state corresponding to some intermediate temperature. Irreversible time evolutions leading to temperature equalization have been obtained by Fannes and Rocca.[4] However, they are not derived from an underlying Hamiltonian evolution. We also remark that in the treatment of Ref. 4 both systems in contact are infinitely extended at the beginning, and therefore have "comparable macroscopic sizes." Instead, in our approach, the "smallness" of the open macroscopic system as compared to the size of the reservoirs is ensured by taking the thermodynamic limit after the limit of weak coupling has been performed on the finite system.[5,6]

In the boson case, it should be possible, in principle, to include condensation in the initial state of the system or of the reservoirs. However, due to the technical difficulties, we do not have complete proofs for this situation. If a system is coupled to a reservoir in a condensed (i.e., nonprimary) state, its reduced dynamics cannot be expected to become Markovian, even in the weak coupling limit. When the initial state of the system shows condensation, and the initial states of the reservoirs do not, the density of the condensate should at most decrease in time. However, in the most obvious model that one can think of (an ideal Bose gas coupled to ideal Bose gases) there is no effective coupling at zero energy: hence the density of the condensate remains constant, and there is no phase transition induced by the dynamics (in contrast to the models of Hepp and Lieb[7] and of Martin[5,6]).

The structure of the paper is as follows. In Sections 2–5 we deal with boson systems and translationally invariant couplings. Section 2 is a summary of some properties of the completely positive, quasi-free maps on the CCR algebra. Section 3 is an extension of Davies' techniques to deal with the weak coupling limit in the presence of several reservoirs at different temperatures. In Section 4 we study the thermodynamic limit, and give some arguments concerning the possible evolution of the condensed phase. In Section 5 we derive the Onsager relations for the heat and matter flows and the properties of entropy production. Finally, in Section 6 we sketch how the preceding results can be obtained also in the fermion case, and we discuss a lattice system for which the coupling to the reservoirs is not translationally invariant.

2. PRELIMINARIES

We give here a brief summary, without proofs, of those parts of the theory of quasi-free, completely positive maps on the CCR algebra that are relevant to this work.[8–13] Let $\mathscr{H}$ be a complex Hilbert space. A representation of the canonical commutation relations (CCR) over $\mathscr{H}$ is a map $h \to W(h)$ of $\mathscr{H}$ into the unitary operators on a Hilbert space $\mathscr{H}_W$ satisfying

$$W(h)W(h') = W(h + h') \exp[\tfrac{1}{2}i(h, h')] \tag{2.1}$$

for all h, h' in $\mathscr{H}$. By Slawny's theorem, all representations of the CCR over a given Hilbert space $\mathscr{H}$ generate isomorphic C^*-algebras. So we can speak of the CCR algebra over $\mathscr{H}$, and denote it by $W(\mathscr{H})$. A state on $W(\mathscr{H})$ is completely determined by its values on the Weyl operators $W(h)$, $h \in \mathscr{H}$. A state ω is called continuous if the map $h \to \omega(W(h))$ is continuous. If B is a positive bounded operator on $\mathscr{H}$, let ω_B be the (continuous) state defined by

$$\omega_B(W(h)) = \exp[-\tfrac{1}{4}(h, [1 + 2B]h)] \tag{2.2}$$

This class of states is a subclass of the quasi-free states. If U is a unitary operator on $\mathscr{H}$, the map α_U on $W(\mathscr{H})$ defined by

$$\alpha_U(W(h)) = W(Uh) \tag{2.3}$$

is a *-automorphism of $W(\mathscr{H})$. Corresponding to each state ω_B and each contraction T on $\mathscr{H}$ there is a completely positive, identity-preserving map $\Phi_{B,T}$ on $W(h)$ satisfying[8,9]

$$\Phi_{B,T}(W(h)) = W(Th)\omega_B(W[(1 - T^*T)^{1/2}h]) \tag{2.4}$$

In particular, when T is a projection, (2.4) is a conditional expectation. If $\{T_t; t \geqslant 0\}$ is a contraction semigroup commuting with B, then $\{\Phi_{B,T_t}; t \geqslant 0\}$ is a semigroup, ω_B is a stationary state for it, and $\lim_{t\to\infty}\omega(\Phi_{B,T_t}(W(h))) = \omega_B(W(h))$ for all h in $\mathscr{H}$ and for all continuous states ω on $W(\mathscr{H})$ if and only if T_t converges strongly to zero as $t \to \infty$.[8,10,13] In the cases we shall be considering, T_t will be of the form $\exp(Kt)$, where K is a bounded operator and $G = -K - K^* \geqslant 0$. In this case, (2.4) can be rewritten as

$$\Phi_{B,T_t}(W(h)) = W(T_t h) \exp\left[-\tfrac{1}{4}\int_0^t ds\, (T_s h, (1 + 2B)GT_s h)\right] \tag{2.5}$$

More generally, if C is a bounded operator satisfying $G \leqslant C \leqslant \alpha G$ for some positive α, then $\{\Phi_t; t \geqslant 0\}$ with

$$\Phi_t(W(h)) = W(T_t h) \exp\left[-\tfrac{1}{4}\int_0^t ds\,(T_s h, CT_s h)\right] \tag{2.6}$$

is a completely positive semigroup[10,12] of which (2.5) is a special case, with $C = (1 + 2B)G$. If B_j, $j = 1,\ldots, r$, are positive bounded operators, K_j are bounded operators with $G_j = -K_j - K_j^* \geqslant 0$, $T_t^j = \exp(K_j t)$, $[B_j, K_j] = 0$ for all j, then Φ_t defined by (2.6) with $T_t = \exp(\sum_{j=1}^r K_j t)$ and $C = \sum_{j=1}^r (1 + 2B_j)G_j$ satisfies

$$\frac{d}{dt}\,\omega(\Phi_t(W(h)))\bigg|_{t=0} = \sum_{j=1}^r \frac{d}{dt}\,\omega(\Phi_{B_j,T_t^j}(W(h)))\bigg|_{t=0} \tag{2.7}$$

for all h in $\mathscr{H}$ and for all continuous states ω on $W(\mathscr{H})$.

3. WEAK COUPLING LIMIT

Here we study the reduced dynamics of a spatially confined boson system S coupled to several boson reservoirs R_j at different temperatures in the interaction picture and in the limit of weak coupling. The case of one reservoir has been treated by Davies in Ref. 2. We use Davies' technique[3] on the test function space. Let $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_r$ be a (separable) Hilbert space, and denote by P_j the projection of $\mathscr{H}$ onto $\mathscr{H}_j$, $j = 0, 1,\ldots, r$. Let H_j, $j = 0, 1,\ldots, r$, be self-adjoint operators in $\mathscr{H}_j$, V_j bounded operators from $\mathscr{H}_0$ into $\mathscr{H}_j$, $j = 1,\ldots, r$, and $V = \sum_{j=1}^r V_j$. For all $\lambda \in [0, \lambda_0)$ define

$$U_t^\lambda = \exp(iZ_\lambda t), \qquad Z_\lambda = H_0 + H_1 + \cdots + H_r + \lambda(V + V^*) \tag{3.1}$$

The system S is described by the algebra of observables $W(\mathscr{H}_0)$ with free evolution determined by the one-particle Hamiltonian H_0 according to $W(h) \to W(\exp(iH_0 t)\, h)$. The algebra of observables of the jth reservoir is $W(\mathscr{H}_j)$ with free evolution given by $W(h) \to W(\exp(iH_j t)\, h)$ and the coupled dynamics of the system plus reservoirs is given by $W(h) \to W(U_t^\lambda h)$, $h \in \mathscr{H}$. Since S is spatially confined, we assume:

(A) H_0 has a pure point spectrum.

We also need the following technical condition:

(B) $\int_0^\infty dt\, \|V^* U_t^0 V\|$ exists.

In order to recover for the reduced dynamics of S the ordinary interaction picture, we have to study the limit as $\lambda \to 0$ of $X_t^\lambda h$, $h \in \mathscr{H}_0$, where

$$X_t^\lambda = P_0 U^\lambda_{t/\lambda^2} U^0_{-t/\lambda^2} \tag{3.2}$$

instead of the limit of $Y_t^\lambda h$,

$$Y_t^\lambda = U^0_{-t/\lambda^2} P_0 U^\lambda_{t/\lambda^2}$$

as in Ref. 3.

Lemma 3.1 (compare Ref. 3). Assuming that conditions (A) and (B) hold, let

$$K = -\sum_k \int_0^\infty dt\, e^{-i\epsilon_k t} E_k V^* U_t^0 E_k \tag{3.3}$$

where the E_k are the spectral projections of H_0 corresponding to distinct eigenvalues ϵ_k, and let T_t be the contraction semigroup on $\mathscr{H}_0$ with generator K. Then

$$\lim_{\lambda \to 0} \| X_t^\lambda h - T_t h \| = 0 \tag{3.4}$$

for all h in $\mathscr{H}_0$, uniformly on bounded intervals in t.

Proof. In Ref. 3 it is shown that $\lim_{\lambda\to 0} \| Y_t^\lambda h - T_t h\| = 0$. Since $X_t^\lambda = U^0_{t/\lambda^2} Y_t^\lambda U^0_{-t/\lambda^2}$ and T_t commutes with U^0_{t/λ^2}, we have

$$\begin{aligned} \|X_t^\lambda h - T_t h\| &= \| Y_t^\lambda U^0_{-t/\lambda^2} h - U^0_{-t/\lambda^2} T_t h \| \\ &= \|(Y_t^\lambda - T_t) U^0_{-t/\lambda^2} h\| \end{aligned}$$

Then, by Davies' result, (3.4) holds for those h that are eigenvectors of H_0. By condition (A), the finite linear combinations of such eigenvectors are dense in $\mathscr{H}_0$. Then (3.4) holds for all h in $\mathscr{H}_0$, since $\|X_t^\lambda - T_t\| \leqslant 2$ for all t and for all λ.

Remark. Note that K has the form $K = \sum_{j=1}^r K_j$, where K_j is given by (3.3) replacing V by V_j. We shall use the notation G_j for $-K_j - K_j^*$.

Corollary 3.2. Let $\{h_n\}$ be a complete orthonormal set (c.o.n.s.) in $\mathscr{H}_0$, let f_{jn} be vectors in $\mathscr{H}_j$ such that

$$(f_{im}, U_t^0 f_{jn}) = \delta_{ij} \delta_{mn} g_j(t) \in L^1(\mathbb{R}) \tag{3.5}$$

and let

$$V_j = \sum_n f_{jn} \otimes \bar{h}_n; \qquad j = 1,\ldots, r \tag{3.6}$$

Then, each V_j is bounded $[\|V_j\|^2 = g_j(0)]$, condition (B) is satisfied, and Lemma (3.1) yields

$$K = \sum_{j=1}^r K_j = \sum_{j=1}^r \left[-\int_0^\infty dt\, g_j(t) \exp(-iH_0 t) \right] \tag{3.7}$$

independently of the particular c.o.n.s. chosen.

We now specify the reference states of the reservoirs to be quasi-free states ω_{B_j} on $W(\mathscr{H}_j)$, $j = 1,\ldots, r$, where the following holds:

(C) $B_j = b_j(H_j)$, the b_j being positive, continuous functions vanishing at infinity.

For instance, if the spectrum of each H_j is $[0,\infty)$, the functions b_j can be taken to be of the form

$$b_j(\epsilon) = \{\exp[\beta(\epsilon - \mu_j)] - 1\}^{-1} \qquad \text{for } \epsilon \geqslant 0 \quad (\beta_j > 0, \mu_j < 0) \tag{3.8}$$

and suitably continued to the negative half-axis. Then, the reduced dynamics on $W(\mathscr{H}_0)$, in the interaction picture, is given by

$$\Phi_{B_1\oplus\cdots\oplus B_r,P_0}\alpha_{U^{\lambda}_{t/\lambda^2}}\alpha_{U^0_{-t/\lambda^2}}[W(h)]$$
$$= W(X_t{}^{\lambda}h)\exp\left\{-\frac{1}{4}\sum_{j=1}^{r}(P_jU^{\lambda}_{t/\lambda^2}U^0_{-t/\lambda^2}h,\,[1 + 2B_j]P_jU^{\lambda}_{t/\lambda^2}U^0_{-t/\lambda^2}h)\right\} \tag{3.9}$$

We show that this tends to a dynamical semigroup of the form (2.6) in the weak coupling limit.

Theorem 3.3. Under assumptions (A)–(C), for all continuous states ω on $W(\mathscr{H}_0)$ and for all h in $\mathscr{H}_0$,

$$\lim_{\lambda\to 0}\omega(\Phi_{B_1\oplus\cdots\oplus B_r,P_0}\alpha_{U^{\lambda}_{t/\lambda^2}}\alpha_{U^0_{-t/\lambda^2}}[W(h)])$$
$$= \omega(W(T_th))\exp\left\{-\frac{1}{4}\sum_{j=1}^{r}\int_0^t ds\,(T_sh,\,[1 + 2b_j(H_0)]G_jT_sh)\right\} \tag{3.10}$$

uniformly on bounded intervals in t.

Proof. For all continuous states ω on $W(\mathscr{H}_0)$ and for all h in $\mathscr{H}_0$

$$\lim_{\lambda\to 0}\omega(W(X_t{}^{\lambda}h)) = \omega(W(T_th))$$

uniformly on bounded intervals in t, by Lemma 3.1. To complete the proof, we show that

$$\lim_{\lambda\to 0}(P_jU^{\lambda}_{t/\lambda^2}U^0_{-t/\lambda^2}h,\,[1 + 2B_j]P_jU^{\lambda}_{t/\lambda^2}U^0_{-t/\lambda^2}h)$$
$$= \int_0^t ds\,(T_sh,\,[1 + 2b_j(H_0)]G_jT_sh) \tag{3.11}$$

uniformly on bounded intervals in t, for all h in $\mathscr{H}_0$ and for all $j = 1,\ldots, r$. Since all the relevant operators are bounded uniformly in λ and uniformly on bounded intervals in t, it suffices to prove (3.11) when h is a finite linear combination of eigenvectors of H_0. Let e_k, e_l be eigenvectors of H_0 corresponding to the eigenvalues ϵ_k, ϵ_l, respectively. We must prove that the expression

$$e^{it(\epsilon_k-\epsilon_l)/\lambda^2}(P_j U^{\lambda}_{t/\lambda^2} e_k, [1 + 2B_j] P_j U^{\lambda}_{t/\lambda^2} e_l) \tag{3.12}$$

tends, as $\lambda \to 0$, uniformly on bounded intervals in t, to the expression

$$\delta_{\epsilon_k,\epsilon_l}[1 + 2b_j(\epsilon_l)] \int_0^t ds\, (T_s e_k, G_j T_s e_l) \tag{3.13}$$

First of all we show that the difference between (3.12) and the expression

$$e^{it(\epsilon_k-\epsilon_l)/\lambda^2}[1 + 2b_j(\epsilon_k)^{1/2} b_j(\epsilon_l)^{1/2}](P_j U^0_{-t/\lambda^2} U^{\lambda}_{t/\lambda^2} e_k, P_j U^0_{-t/\lambda^2} U^{\lambda}_{t/\lambda^2} e_l) \tag{3.12'}$$

vanishes as $\lambda \to 0$, uniformly in t. Indeed,

$$B_j^{1/2} P_j = b_j(H_j)^{1/2} P_j = b_j(Z_0)^{1/2} P_j = P_j b_j(Z_0)^{1/2}$$

and

$$\|[b_j(Z_0)^{1/2}, U^{\lambda}_{t/\lambda^2}]\| \leqslant 2\|b_j(Z_0)^{1/2} - b_j(Z_\lambda)^{1/2}\| \to 0 \qquad \text{as} \quad \lambda \to 0$$

since $b_j(Z_\lambda)^{1/2}$ commutes with $U^{\lambda}_{t/\lambda^2}$ and Z_λ tends to Z_0 in the norm resolvent sense (use, e.g., Theorem IV.3.11 of Ref. 14 and Theorem VIII.20 of Ref. 15). Finally, $b_j(Z_0)^{1/2} e_k = b_j(\epsilon_k)^{1/2} e_k$ and U^0_{-t/λ^2} is a unitary operator commuting with P_j. Using

$$U^0_{-t/\lambda^2} U^{\lambda}_{t/\lambda^2} = 1 + i\lambda \int_0^{t/\lambda^2} ds\, U^0_{-s}(V + V^*) U_s^{\lambda}$$

recalling that $P_j(V + V^*) = P_j V_j = P_j V_j V_0$, and performing some change of variables in the integration, we find that the scalar product in (3.12′) is given by

$$\begin{aligned} &\int_0^t ds \int_0^{(t-s)/\lambda^2} du\, [(U^{\lambda}_{(s/\lambda^2)+u} e_k, V_j^* U_u{}^0 V_j U^{\lambda}_{s/\lambda^2} e_l) \\ &\qquad + (V_j^* U_u{}^0 V_j U^{\lambda}_{s/\lambda^2} e_k, U^{\lambda}_{(s/\lambda^2)+u} e_l)] \\ &= \int_0^t ds \int_0^{(t-s)/\lambda^2} du\, [(Y^{\lambda}_{s+\lambda^2 u} e_k, U^0_{-(s/\lambda^2)-u} V_j^* U_u{}^0 V_j U^0_{s/\lambda^2} Y_s{}^{\lambda} e_l) \\ &\qquad + (U^0_{-(s/\lambda^2)-u} V_j^* U_u{}^0 V_j U^0_{s/\lambda^2} Y_s{}^{\lambda} e_k, Y^{\lambda}_{s+\lambda^2 u} e_l)] \end{aligned}$$

Now define functions $F^{\lambda,t}_{kl}(s, u)$ by

$$F^{\lambda,t}_{kl}(s, u) = \begin{cases} e^{is(\epsilon_k-\epsilon_l)/\lambda^2}(Y^{\lambda}_{s+\lambda^2 u} e_k, U^0_{-(s/\lambda^2)-u} V_j^* U_u{}^0 V_j U^0_{s/\lambda^2} Y_s{}^{\lambda} e_l) \\ \qquad \text{for} \quad u \leqslant (t - s)/\lambda^2 \\ 0 \qquad \text{for} \quad u > (t - s)/\lambda^2 \end{cases} \tag{3.14}$$

Then, (3.12′) can be written as

$$\int_0^t ds \int_0^\infty du\, e^{i(t-s)(\epsilon_k-\epsilon_l)/\lambda^2}[1 + 2b_j(\epsilon_k)^{1/2}b_j(\epsilon_l)^{1/2}][F_{kl}^{\lambda,t}(s,u) + \bar{F}_{lk}^{\lambda,t}(s,u)]$$

Since $Y^\lambda_{s+\lambda^2 u}$ and $Y_s{}^\lambda$ tend strongly to T_s as $\lambda \to 0$,[3] and all the operators involved in (3.14) are bounded by 1, uniformly in λ, t, s, and u, the limit as $\lambda \to 0$ of $F_{kl}^{\lambda,t}(s,u)$ is

$$F_{kl}(s,u) = (T_s e_k, U^0_{-u} V_j{}^* U_u{}^0 V_j T_s e_l) \qquad \text{for all } s \in [0,t), \quad u \in [0,\infty)$$

Now, recalling the explicit expression of K_j, we see that (3.13) is the limit as $\lambda \to 0$ of

$$\int_0^t ds \int_0^\infty du\, e^{i(t-s)(\epsilon_k-\epsilon_l)/\lambda^2}[1 + 2b_j(\epsilon_k)^{1/2}b_j(\epsilon_l)^{1/2}][F_{kl}(s,u) + \bar{F}_{lk}(s,u)] \tag{3.13′}$$

[when $\epsilon_k \neq \epsilon_l$ the limit as $\lambda \to 0$ of (3.13′) can be shown to vanish by an integration by parts]. Thus

$$\begin{aligned}
&\lim_{\lambda\to 0}|(3.12) - (3.13)| \\
&\quad = \lim_{\lambda\to 0}|(3.12′) - (3.13′)| \\
&\quad \leqslant \lim_{\lambda\to 0}\, [1 + 2b_j(\epsilon_k)^{1/2}b_j(\epsilon_l)^{1/2}] \int_0^t ds \int_0^\infty du \\
&\quad\quad \times \{|F_{kl}^{\lambda,t}(s,u) - F_{kl}(s,u)| + |\bar{F}_{lk}^{\lambda,t}(s,u) - \bar{F}_{lk}(s,u)|\}
\end{aligned} \tag{3.15}$$

The integrand in (3.15) is bounded by the integrable function $4\|V_j{}^* U_u{}^0 V_j\|$; hence, by the dominated convergence theorem, the limit (3.15) is zero. Uniformness of convergence on bounded intervals in t clearly holds. This proves the theorem.

Comment. It is clear from the discussion at the end of Section 2 that, loosely speaking, the dynamical semigroup obtained in the weak coupling limit has a generator which is the sum of the generators for the dynamics that would have been obtained by coupling the system to the jth reservoir alone, $j = 1, \ldots, r$ (for a precise discussion of the meaning of "generator" of a quasi-free dynamical semigroup on the CCR algebra see Ref. 12).

A stationary state for this dynamical semigroup is

$$\omega_{B_\infty} = \lim_{t\to\infty} \omega_0 \circ \Phi_t$$

where

$$B_\infty = \int_0^\infty dt\, T_t{}^* \sum_{j=1}^r G_j b_j(H_0) T_t \tag{3.16}$$

With a coupling described as in Corollary 3.2, B_∞ is given simply by

$$B_\infty = b(H_0) \equiv \mathrm{B}$$

$$b(\epsilon) = \left[\sum_{j=1}^{r} \hat{g}_j(\epsilon)\right]^{-1} \sum_{j=1}^{r} \hat{g}_j(\epsilon) b_j(\epsilon) \tag{3.17}$$

where

$$\hat{g}_j(\epsilon) = \int_{-\infty}^{\infty} dt\, e^{-i\epsilon t} g_j(t) \tag{3.18}$$

is a nonnegative continuous function vanishing at infinity, for all $j = 1,\ldots, r$, and Φ_t is of the type Φ_{B,T_t}. When the functions b_j describe thermal states, ω_B is not a thermal state at some intermediate temperature and chemical potential; rather, its two-point functions are a sort of weighted average of the two-point functions of the thermal states corresponding to thermal equilibrium with each reservoir separately.

4. THERMODYNAMIC LIMIT

Let $\{\mathscr{H}_\Lambda\}$ be a directed net of Hilbert spaces, indexed by an increasing family $\{\Lambda\}$ of bounded regions of space ($\mathbb{R}^\nu$ or $\mathbb{Z}^\nu$) whose union is the whole space, and let $\mathscr{H}$ be the completion of $\bigcup_\Lambda \mathscr{H}_\Lambda$. The algebra of observables for the system confined in the region Λ is $W(\mathscr{H}_\Lambda)$. Notice that the norm closure of $\bigcup_\Lambda W(\mathscr{H}_\Lambda)$ is properly contained in $W(\mathscr{H})$. Concerning the dynamics, we shall make the following assumption:

(D) For all Λ there is a self-adjoint operator H_Λ in $\mathscr{H}_\Lambda$ with pure point spectrum, and there exists a dense domain $\mathscr{D}$ in $\mathscr{H}$ such that (i) each $h \in \mathscr{D}$ is in the domain of H_Λ for large enough Λ; (ii) for all $h \in \mathscr{D}$, $\lim_{\Lambda\nearrow\infty} H_\Lambda h = Hh$ exists; (iii) H is essentially self-adjoint on $\mathscr{D}$.

For all Λ it is possible to derive a dynamical semigroup Φ_t^Λ of $W(\mathscr{H}_\Lambda)$ by coupling the system to suitable reservoirs with an interaction $\lambda(V_\Lambda + V_\Lambda^*)$ and taking the weak coupling limit, as in Theorem 3.3. In general, Φ_t^Λ will depend on Λ both through V_Λ and H_Λ, and this can in principle give rise to a complicated structure, for which the limit $\Lambda \nearrow \infty$ cannot be easily taken. So we make a simplifying assumption on V_Λ:

(E) $V_\Lambda = \sum_{j=1}^{r} V_j^\Lambda$, where r is independent of j, V_j^Λ is of the form (3.6), and the functions $g_j^{(t)}$ of (3.5) are independent of Λ.

Under assumption (E), the semigroup Φ_t^Λ of Theorem 3.3 is of the form

$$\Phi_t^\Lambda = \Phi_{B_\Lambda, T_t^\Lambda} \tag{4.1}$$

where

$$B_\Lambda = b(H_\Lambda) \qquad [b \text{ defined by Eq. (3.17)}] \tag{4.2}$$

$$T_t^\Lambda = \exp(K_\Lambda t), \qquad K_\Lambda = -\sum_{j=1}^{r} \int_0^\infty dt\, g_j(t) \exp(-iH_\Lambda t) \tag{4.3}$$

Then we have the following result.

Theorem 4.1. Let $B_\Lambda = b(H_\Lambda)$ and $T_t^\Lambda = \exp[k(H_\Lambda)t]$, where $b(\cdot)$ and $k(\cdot)$ are continuous bounded functions, independent of Λ, with $b \geqslant 0$ $-k - \bar{k} \geqslant 0$. Assume that (D) holds and let

$$B = b(H), \qquad T_t = \exp[k(H)t] \tag{4.4}$$

Then

$$\lim_{\Lambda \nearrow \infty} \omega(\Phi_{B_\Lambda, T_t^\Lambda}(W(h))) = \omega(\Phi_{B, T_t}(W(h))) \tag{4.5}$$

uniformly on bounded intervals in t, for all h in $\bigcup_\Lambda \mathscr{H}_\Lambda$ and for all states that can be extended to continuous states on $W(\mathscr{H})$.

Proof. It is clear from formula (2.4) that it suffices to prove that T_t^Λ converges strongly to T_t and that $[1 - (T_t^\Lambda)^* T_t^\Lambda] B_\Lambda$ converges weakly to $(1 - T_t^* T_t)B$, uniformly on bounded intervals in t. By assumption (D) and Corollary VIII.1.6 of Ref. 14, H_Λ converges to H in the strong resolvent sense. Hence, by Theorem VIII.20 of Ref. 15, B_Λ converges strongly to B and $[1 - k(H_\Lambda)]^{-1}$ converges strongly to $[1 - k(H)]^{-1}$. Thus, by Theorem IX.2.16 of Ref. 14, T_t^Λ converges strongly to T_t, uniformly on bounded intervals in t. The same holds for $[1 - (T_t^\Lambda)^* T_t^\Lambda]$.

As an example, we consider the ideal Bose gas.[16] Let Λ denote a bounded region of $\mathbb{R}^3$, let $\mathscr{H}_\Lambda = L^2(\Lambda)$, and let Δ_Λ be one of the self-adjoint extensions of the Laplacian on $C_0^\infty(\Lambda)$. Define the local Hamiltonian H_Λ as $-\frac{1}{2}\Delta_\Lambda$; H has pure point spectrum, with eigenvalues ϵ_k^Λ and eigenvectors e_k^Λ ($k = 1, 2, \ldots$; $\epsilon_k^\Lambda \geqslant \epsilon_{k-1}^\Lambda$). Assumption (D) is satisfied with $\mathscr{D} = C_0^\infty(\mathbb{R}^3) = \bigcup_\Lambda C_0^\infty(\Lambda)$. The algebra of observables for an ideal Bose gas confined in the box Λ is $W(\mathscr{H}_\Lambda)$, and the free dynamics is given by $\alpha_{\exp(iH_\Lambda t)}$. We couple this system to several thermal reservoirs, which are assumed to be infinitely extended ideal Bose gases, in quasi-free states determined by functions

$$b_j(\epsilon) = [\exp \beta_j(\epsilon - \mu_j) - 1]^{-1} \qquad \text{for} \quad \epsilon \in [0, \infty) \tag{4.6}$$

with $\beta_j > 0$, $\mu_j < 0$, $j = 1, \ldots, r$. Choose the coupling to satisfy (E). Then (4.1)–(4.3) hold and Theorem 4.1 can be applied to study the thermodynamic limit $\Lambda \nearrow \infty$.

However, we remark that continuous states on $W(L^2(\mathbb{R}^3))$ are not exhaustive of the states of interest. To see this, it suffices to consider the limit as $\Lambda \nearrow \infty$ of the grand canonical state

$$\omega_{Q^{\Lambda}_{\beta,\rho}}, \qquad Q^{\Lambda}_{\beta,\rho} = [\exp \beta(H_{\Lambda} - \mu_{\Lambda,\beta,\rho}) - 1]^{-1}$$

keeping the inverse temperature β and the mean density ρ fixed.[16] This limit defines a state on $\overline{\bigcup_{\Lambda} W(\mathscr{H}_{\Lambda})}$. However, only when ρ is less than the critical density for condensation $\rho < \rho_c = (2\pi\beta)^{-3/2}\sum_{n=1}^{\infty} n^{-3/2}$ can this state be extended to a continuous (quasi-free) state on $W(L^2(\mathbb{R}^3))$, of the form

$$\omega_{Q_{\beta,\rho}}, \qquad Q_{\beta,\rho} = [\exp \beta(H - \mu_{\beta,\rho}) - 1]^{-1}, \qquad \mu_{\beta,\rho} < 1$$

On the other hand, when $\rho \geqslant \rho_c$, the limit state is given by

$$\omega_{\mathscr{Q}_{\beta,\rho}}(W(h)) = \exp[-\tfrac{1}{4}\|h\|^2 - \tfrac{1}{2}\mathscr{Q}_{\beta,\rho}(h, h)]$$

where

$$\mathscr{Q}_{\beta,\rho}(h, h) = (\rho - \rho_c)|e_1^{\Lambda^1}(0)|^2|\hat{h}(0)|^2 + (h, [\exp(\beta H) - 1]^{-1}h) \tag{4.7}$$

$e_1^{\Lambda^1}$ being the eigenvector corresponding to the lowest eigenvalue for H_{Λ} in a region Λ^1 of unit volume; see Ref. 16 for the details. The maximal domain of the quadratic form (4.7) is $L^1(\mathbb{R}^3) \bigcap L^2(\mathbb{R}^3)$. If $\rho < \rho_c$, $Q^{\Lambda}_{\beta,\rho}$ converges weakly to $Q_{\beta,\rho}$. Furthermore,

$$\lim_{\Lambda \nearrow \infty} \omega_{Q^{\Lambda}_{\beta,\rho}}(\Phi_{B_{\Lambda},T_t^{\Lambda}}(W(h))) = \omega_{Q_{\beta,\rho}}(\Phi_{B,T_t}(W(h)))$$

(combine weak converge of $Q^{\Lambda}_{\beta,\rho}$ to $Q_{\beta,\rho}$ with strong convergence of T_t^{Λ} to T_t). When $\rho \geqslant \rho_c$, we conjecture that

$$\begin{aligned}
&\lim_{\Lambda \nearrow \infty} \omega_{Q^{\Lambda}_{\beta,\rho}}(\Phi_{B_{\Lambda},T_t^{\Lambda}}(W(h))) \\
&= \omega_{\mathscr{Q}_{\beta,\rho}}(\Phi_{B,T_t}(W(h))) \\
&= \exp[-\tfrac{1}{4}\|h\|^2 - \tfrac{1}{2}(\rho - \rho_c)|e_1^{\Lambda^1}(0)|^2|\widehat{T_t h}(0)|^2 \\
&\quad - \tfrac{1}{2}(T_t h, [\exp(\beta H) - 1]^{-1}T_t h) + \tfrac{1}{2}(h, (1 - T_t^* T_t)b(H)h)]
\end{aligned}$$

at least when h is in $\mathscr{S}(\mathbb{R}^3)$ and all functions g_j are C^{∞}, so that also $T_t h$ is in $\mathscr{S}(\mathbb{R}^3)$. However, we have been unable to give a proof of this (surprisingly enough, the unbounded operator part $(h, [\exp(\beta H) - 1]^{-1}h)$ gives more trouble than the singular part, proportional to $|\hat{h}(0)|^2$). If the conjecture is true, then the density of the condensed phase at time t is given by

$$\rho_1(t) = (\rho - \rho_c) \exp\left[-\sum_{j=1}^{r} \hat{g}_j(0)t\right]$$

Because of our technical assumption that the functions g_j are in L^1, and since 0 is a boundary point of the support $\hat{g}_j$, $\hat{g}_j(0)$ actually vanishes, so that the zero-energy mode is decoupled from the reservoir and does not evolve.

5. ONSAGER RELATIONS AND ENTROPY PRODUCTION

In this section, we discuss for our model of an ideal Bose gas those non-equilibrium thermodynamic properties (Onsager relations, positivity of entropy production, principle of minimal entropy production in the linear regime) that were studied by Spohn and Lebowitz[1] for N-level systems. We shall only consider quasi-free states ω_Q, where Q is a function of the Hamiltonian, and dynamical semigroups Φ_{B,T_t} as specified in Eqs. (4.1)–(4.5). The time evolution of $\omega_Q = \omega_{q(H_\Lambda)}$ or $\omega_{q(H)}$ is given by (we drop for the moment the subscript Λ)

$$\omega_{q(H)} \circ \Phi_{B,T_t} = \omega_{q^t(H)} \tag{5.1a}$$

where

$$q^t(\epsilon) = \left\{\exp\left[-\sum_{j=1}^{r} \hat{g}_j(\epsilon)t\right]\right\}q(\epsilon) + \left\{1 - \exp\left[-\sum_{j=1}^{r} \hat{g}_j(\epsilon)t\right]\right\}b(\epsilon) \tag{5.1b}$$

The evolution of $\omega_{q(H)}$ when the system is coupled only to the reservoirs of jth type is $[T_t^j = \exp(K_j t)]$

$$\omega_{q(H)} \circ \Phi_{B_j,T_t^j} = \omega_{q_j^t(H)} \tag{5.2a}$$

where

$$q_j^t(\epsilon) = \{\exp[-\hat{g}_j(\epsilon)t]\}q(\epsilon) - \{1 - \exp[-\hat{g}_j(\epsilon)t]\}b(\epsilon) \tag{5.2b}$$

Let also $\dot{q}^t(\epsilon) = dq^t(\epsilon)/dt$, $\dot{q}_j^t(\epsilon) = dq_j^t(\epsilon)/dt$. Then

$$\dot{q}^{t=0}(\epsilon) = \sum_{j=1}^{r} \dot{q}_j^{t=0}(\epsilon) = \sum_{j=1}^{r} \hat{g}_j(\epsilon)[b_j(\epsilon) - q(\epsilon)] \tag{5.3}$$

We must require some conditions on the function q, which ensure that the quantities we compute are finite in finite volume, and that their densities tend to finite expressions in the thermodynamic limit. In finite volume, the mean energy, the mean particle number, and the entropy for a state ω_Q are given, respectively, by

$$\omega_Q(\tilde{H}_\Lambda) = \operatorname{tr} QH_\Lambda \tag{5.4a}$$

($\tilde{H}_\Lambda$ is the second quantization of H_Λ),

$$\omega_Q(N_\Lambda) = \operatorname{tr} Q \tag{5.4b}$$

(N_Λ is the number operator, i.e., the second quantization of $\mathbf{1}_{\mathscr{H}_\Lambda}$), and[17,18]

$$S(\omega_Q) = \operatorname{tr}[(1 + Q)\log(1 + Q) - Q\log Q] \tag{5.4c}$$

In order to control the infinite-volume limit, we use a theorem of Ref. 17 for the trace of functions of the Laplacian.

Theorem 5.1.[17] Let f: $\mathbb{R}^+ \to \mathbb{R}$ be a bounded, eventually decreasing, differentiable function, with f' in $L^1_{\text{loc}}(\mathbb{R}^+)$. Then $\operatorname{tr} f(-\Delta_\Lambda)$ exists for some region Λ if and only if $\int f(x^2)\, d^3x$ exists. If this holds, then also

$$\lim_{\Lambda \nearrow \infty} |\Lambda|^{-1} \operatorname{tr} f(-\Delta_\Lambda) = \int \frac{d^3x}{(2\pi)^3} f(x^2) \tag{5.5}$$

where $\{\Lambda\}$ is an increasing family of bounded regions containing the origin, of the form $\Lambda_L = \{x \in \mathbb{R}^3 : L^{-1}x \in \Lambda_1\}$, $L \geqslant 1$, and $|\Lambda|$ is the volume of Λ. The Laplacian Δ_Λ is defined with Dirichlet boundary conditions or with boundary conditions $\partial h/\partial n + (a/L)h = 0$, where $a \in \mathbb{R}$ and $\partial/\partial n$ is the directional derivative in the direction of the outward normal n to the boundary $\partial\Lambda_L$.

In order to apply the above theorem, and to ensure the finiteness of the expressions (5.4a)–(5.4c) at all times under the evolution given by (5.1) and (5.2), we need the following conditions on the function q.

(F) q is positive, bounded, and bounded away from zero in any bounded subset of $\mathbb{R}^+$; q is differentiable and q' is in $L^1_{\text{loc}}(\mathbb{R}^+)$; the functions $q(\epsilon)$, $\epsilon q(\epsilon)$, and $-b_j(\epsilon) \log q(\epsilon) (j = 1,..., r)$, where b_j is given by (4.6), are eventually decreasing; the following integrals exist: (i) $\int x^2 q(x^2)\, d^3x$, (ii) $\int q(x^2)\, d^3x$, (iii) $-\int q(x^2) \log q(x^2)\, d^3x$, (iv) $-\int b_j(x^2) \log q(x^2)\, d^3x\ (j = 1,..., r)$.

Moreover, we have to assume that the functions g_j $(j = 1,..., r)$ are differentiable, with bounded derivative, and eventually decreasing. Remark that all the functions b_j satisfy (F). By Theorem 5.1, conditions (i)–(iv) imply their finite-volume counterparts: (i') $\sum_k \epsilon_k{}^\Lambda q(\epsilon_k{}^\Lambda) < \infty$, etc.; this therefore ensures the finiteness of (5.4a)–(5.4c) in finite volume. The function b given by (3.17) satisfies (i)–(iv), as well as their finite-volume counterparts, and if q satisfies (i)–(iv), also q^t and $q_j{}^t$ satisfy (i)–(iv) and their finite-volume counterparts for all $t \geqslant 0$. Indeed,

$$q + b \geqslant q^t \geqslant e^{-\gamma t} q$$

where

$$\gamma = \max_j \sup_\epsilon \hat{g}_j(\epsilon) \tag{5.6}$$

[$0 < \gamma < \infty$, since the functions $\hat{g}_j(\epsilon)$ are positive and continuous, and vanish at 0 and ∞]. From the first inequality, it follows that (i) and (ii) for q imply (i) and (ii) for q^t, $q_j{}^t$. From the second inequality, we have

$$-\log q_t \leqslant \gamma t - \log q$$

so that (ii) and (iii) for q imply (iii) for q^t, $q_j{}^t$ [and similarly for (iv)].

Lemma 5.2. The heat and matter flows coming into the system in the state $\omega_{q(H_\Lambda)}$ from the reservoirs of jth type are given, respectively, by

$$J_j^H(q(H_\Lambda)) = \frac{d}{dt}\omega_{q_j^t(H_\Lambda)}(\tilde{H}_\Lambda)|_{t=0}$$

$$= \sum_{k=1}^{\infty} \epsilon_k^\Lambda \hat{g}_j(\epsilon_k^\Lambda)[b_j(\epsilon_k^\Lambda) - q(\epsilon_k^\Lambda)] \tag{5.7a}$$

and

$$J_j^N(q(H_\Lambda)) = \frac{d}{dt}\omega_{q_j^t(H_\Lambda)}(N_\Lambda)|_{t=0}$$

$$= \sum_{k=1}^{\infty} \hat{g}_j(\epsilon_k^\Lambda)[b_j(\epsilon_k^\Lambda) - q(\epsilon_k^\Lambda)] \tag{5.7b}$$

Proof. We prove (5.7a), the argument leading to (5.7b) being the same. By Eq. (5.4a) and assumption (F(i)), $\omega_{q_j^t(H_\Lambda)}(\tilde{H}_\Lambda)$ is finite at all times, and is given by the convergent series $\sum_{k=1}^{\infty}\epsilon_k^\Lambda q_j^t(\epsilon_k^\Lambda)$. Consider the series $\sum_{k=1}^{\infty} \epsilon_k^\Lambda \dot{q}_j^t(\epsilon_k^\Lambda)$ obtained by differentiation term by term. We have $\epsilon|\dot{q}_j^t(\epsilon)| \leqslant \gamma\epsilon e^\gamma[q(\epsilon) + b_j(\epsilon)]$ for $t \leqslant 1$, where γ is given by (5.6), so that the series under consideration is majorized by a convergent series which is independent of t. Hence, it converges uniformly in t and differentiation term by term of $\sum_{k=1}^{\infty}\epsilon_k^\Lambda q_j^t(\epsilon_k^\Lambda)$ is permissible. The result is (5.7a).

Then, the entropy flowing per unit time into the system from the reservoirs of jth type is given by

$$J_j^S(q(H_\Lambda)) = \beta_j J_j^H(q(H_\Lambda)) - \beta_j\mu_j J_j^N(q(H_\Lambda)) \tag{5.8}$$

and the thermodynamic forces conjugated to H and N are

$$f_j^H = \beta_j - \beta_0, \qquad f_j^N = -\beta_j\mu_j + \beta_0\mu_0 \tag{5.9}$$

where β_0 and μ_0 are some intermediate inverse temperature and chemical potential. Let $\{\alpha\} = \{\alpha_1^H,..., \alpha_r^H;\ \alpha_1^N,..., \alpha_r^N\} = (\beta_1,..., \beta_r;\ -\beta_1\mu_1,..., -\beta_r\mu_r)$ and let $\{\alpha_0\}$ be the $2r$-tuple obtained by setting all $\beta_j = \beta_0$ and $\mu_j = \mu_0$. Let also $x_k^H = \epsilon_k^\Lambda$ and $x_k^N = 1$ for all k.

Theorem 5.3. The Onsager relations for the heat and matter flows in the stationary state $\omega_{b(H_\Lambda)}$ hold. Indeed, letting

$$L_{ij}^{mn}(\{\alpha\}) = \frac{\partial}{\partial f_j^n} J_i^m(b(H_\Lambda)); \qquad m, n = H, N; \quad i, j = 1,..., r \tag{5.10}$$

we have

$$L_{ij}^{mn}(\{\alpha_0\}) = \sum_{k=1}^{\infty} \frac{\hat{g}_i(\epsilon_k^\Lambda)\hat{g}_j(\epsilon_k^\Lambda)}{\sum_{l=1}^{r}\hat{g}_l(\epsilon_k^\Lambda)} x_k^m x_k^n \frac{\exp[\beta_0(\epsilon_k^\Lambda - \mu_0)]}{\{\exp[\beta_0(\epsilon_k^\Lambda - \mu_0)] - 1\}^2} \tag{5.11}$$

which is manifestly symmetric under the interchanges $i \rightleftharpoons j$ and $m \rightleftharpoons n$.

Proof. Replacing b for q in expressions (5.7a) and (5.7b) for the J_i^m, we get

$$J_i^m(b(H_\Lambda)) = \sum_{k=1}^{\infty} \sum_{j=1}^{r} x_k{}^m \frac{\hat{g}_i(\epsilon_k{}^\Lambda)\hat{g}_j(\epsilon_k{}^\Lambda)}{\sum_{l=1}\hat{g}_l(\epsilon_k{}^\Lambda)} [b_i(\epsilon_k{}^\Lambda) - b_j(\epsilon_k{}^\Lambda)] \tag{5.12}$$

which converges by Lemma 5.2. Treat β_j and $\beta_j\mu_j$ as independent variables. If we differentiate (5.12) term by term, and then put $\{\alpha\} = \{\alpha_0\}$, we obtain (5.11), so that we only have to prove that the series of the derivatives is uniformly convergent in a neighborhood of $\{\alpha_0\}$. For $\beta_j > \beta_0/2$ and $-\mu_j > 0$ we have

$$(e^{\beta_j(\epsilon-\mu_j)} - 1)^{-1} \leqslant (e^{\beta_j\epsilon} - 1)^{-1} \lessapprox (e^{\beta_0\epsilon/2} - 1)^{-1}$$

and for $\beta_j > \beta_0/2$ and $-\mu_j > -\mu_0/2$ we have

$$\frac{e^{\beta_j(\epsilon-\mu_j)}}{(e^{\beta_j(\epsilon-\mu_j)} - 1)^2} = \frac{1}{e^{\beta_j(\epsilon-\mu_j)} - 1}\left(1 + \frac{1}{e^{\beta_j(\epsilon-\mu_j)} - 1}\right)$$

$$\leqslant \frac{1}{e^{\beta_0\epsilon/2} - 1}\left(1 + \frac{1}{e^{-\beta_0\mu_0/4} - 1}\right)$$

Moreover, since all g are positive,

$$0 \leqslant \frac{\hat{g}_i(\epsilon)\hat{g}_j(\epsilon)}{\sum_l \hat{g}_l(\epsilon)} \leqslant \frac{\hat{g}_i(\epsilon)\hat{g}_j(\epsilon)}{\hat{g}_i(\epsilon)} \leqslant \gamma$$

and the series

$$\sum_{k=1}^{\infty} \gamma x_k{}^m x_k{}^n \frac{1}{\exp(\beta_0\epsilon_k{}^\Lambda/2) - 1}\left[1 + \frac{1}{\exp(-\beta_0\mu_0/4) - 1}\right]$$

converges. This proves the theorem.

As in Ref. 1, we define the entropy production $\sigma(q(H_\Lambda))$ as the source term in the entropy balance equation

$$\frac{d}{dt} S(q^t(H_\Lambda))|_{t=0} = \sum_{j=1}^{r} J_j{}^S(q(H_\Lambda)) + \sigma(q(H_\Lambda)) \tag{5.13}$$

where q^t is given by (5.1b), J^S is given by (5.8), and the expression for $S(q^t(H_\Lambda))$, obtained upon inserting q^t in (5.4c), is

$$S(q^t(H_\Lambda)) = \sum_{k=1}^{\infty} \{[1 + q^t(\epsilon_k{}^\Lambda)] \log[1 + q^t(\epsilon_k{}^\Lambda)] - q^t(\epsilon_k{}^\Lambda) \log q^t(\epsilon_k{}^\Lambda)\} \tag{5.14}$$

which is finite by assumptions (F(ii)) and (F(iii)).

Theorem 5.4. The entropy production $\sigma(q(H_\Lambda))$ is given by

$$\sigma(q(H_\Lambda)) = \sum_{k=1}^{\infty} \sum_{j=1}^{r} \hat{g}_j(\epsilon_k{}^\Lambda)[b_j(\epsilon_k{}^\Lambda) - q(\epsilon_k{}^\Lambda)] \times \{\log[1 + q(\epsilon_k{}^\Lambda)] - \log q(\epsilon_k{}^\Lambda) - \beta_j(\epsilon_k{}^\Lambda - \mu_j)\} \tag{5.15}$$

It is positive and convex as a function of q. The state $\omega_{q_m(H_\Lambda)}$ of minimal entropy production is characterized by the condition

$$\sum_{j=1}^{r} \hat{g}_j(\epsilon)\left[\log\frac{1 + q_m(\epsilon)}{q_m(\epsilon)} - \beta_j(\epsilon - \mu_j) + \frac{b_j(\epsilon) - q_m(\epsilon)}{q_m(\epsilon)[1 + q_m(\epsilon)]}\right] = 0 \tag{5.16}$$

for all $\epsilon = \epsilon_k{}^\Lambda$. The stationary state $\omega_{b(H_\Lambda)}$ does not satisfy (5.16) unless all temperatures and chemical potentials coincide. If β_0 and μ_0 are some intermediate temperature and chemical potential, q_m and b coincide in the linear approximation in $\beta_j - \beta_0$ and $\beta_j\mu_j - \beta_0\mu_0$.

Proof. In order to establish the validity of the expression (5.15), which is obtained by differentiating (5.14) term by term and subtracting the entropy flows (5.8), we must prove the uniform convergence of the series of the derivatives, which is

$$-\sum_{k=1}^{\infty} \{\dot{q}^t(\epsilon_k{}^\Lambda) \log q^t(\epsilon_k{}^\Lambda) - \dot{q}^t(\epsilon_k{}^\Lambda) \log[1 + q^t(\epsilon_k{}^\Lambda)]\}$$

Now $q^t \leqslant q + b$ and $\dot{q}^t \leqslant \gamma(q + b)$ for $t \geqslant 0$. Also, $-\log q^t(\epsilon) \leqslant -\gamma - \log q(\epsilon)$ for $t \in [0, 1]$. Therefore, by (F(ii)–(iv)) the series converges uniformly. The convexity of the function

$$q \to \tilde{\sigma}(q) = \hat{g}[b - q]\log[(1 + q)/q] \qquad (\hat{g} \geqslant 0; \quad q, b > 0)$$

is shown by computing its second derivative, which is

$$\hat{g}[q + b(1 + 2q)]/[q^2(1 + q)^2] \geqslant 0$$

Thus, for $0 < \alpha < 1$, we have (compare Ref. 19) $\tilde{\sigma}(\alpha q + (1 - \alpha)b) \leqslant \tilde{\alpha}\sigma(q)$ since $\tilde{\sigma}(b) = 0$. Hence

$$\begin{aligned} 0 &\leqslant \tilde{\sigma}(q) - \alpha^{-1}\tilde{\sigma}(\alpha q + (1 - \alpha)b) \\ &= \hat{g}[b - q]\left[\log\frac{1 + q}{q} - \log\frac{1 + \alpha q + (1 - \alpha)b}{\alpha q + (1 - \alpha)b}\right] \end{aligned}$$

In the limit $\alpha \to 0$,

$$\hat{g}[b - q]\left[\log\frac{1 + q}{q} - \log\frac{1 + b}{b}\right] \geqslant 0$$

Now $\beta_j(\epsilon - \mu_j) = \log\{[1 + b_j(\epsilon)]/b_j(\epsilon)\}$. Thus, each summand of (5.15) is positive and convex as a function of q. In order to minimize (5.15), we can therefore minimize each summand independently, differentiating with respect to q. Direct computation leads to condition (5.16). This is not satisfied by the function b, which gives the stationary state: insertion of (3.17) into the l.h.s. of (5.16) gives

$$\sum_{j=1}^{r} \hat{g}_j(\epsilon)\left[\log\frac{1+b(\epsilon)}{b(\epsilon)} - \log\frac{1+b_j(\epsilon)}{b_j(\epsilon)}\right]$$

which is not zero unless all inverse temperatures and chemical potentials are the same. However, b and q_m can be regarded as functions of $\{\alpha\} = (\beta_1,\ldots, \beta_r;\ -\beta_1\mu_1,\ldots, -\beta_r\mu_r)$ and expanded about $\{\alpha_0\}$. The zeroth-order term is $\{\exp[\beta_0(\epsilon - \mu_0)] - 1\}^{-1}$ for both. A straightforward computation yields also

$$\begin{aligned}\frac{\partial}{\partial\alpha_j^n} q_m(\epsilon)\big|_{\{\alpha\}=\{\alpha_0\}} &= -\left[\sum_{l=1}^{r} \hat{g}_l(\epsilon)\right]^{-1} \hat{g}_j(\epsilon)x^n(\epsilon)\frac{e^{\beta_0(\epsilon-\mu_0)}}{(e^{\beta_0(\epsilon-\mu_0)}-1)^2}\\ &= \frac{\partial}{\partial\alpha_j^n} b(\epsilon)\bigg|_{\{\alpha\}=\{\alpha\}_0}; \qquad j = 1,\ldots, r; \quad n = H, N\end{aligned} \tag{5.17}$$

where $x^H(\epsilon) = \epsilon$ and $x^N(\epsilon) = 1$; thus the principle of minimal entropy production holds in the linear approximation.

In the limit of large volume, the heat and matter flows and the entropy production grow with $|\Lambda|$. So we divide by $|\Lambda|$ and take the limit $|\Lambda| \to \infty$, obtaining heat flow densities $\bar{J}_j^H(q(H))$, matter flow densities $\bar{J}_j^N(q(H))$, and entropy production density $\bar{\sigma}(q(H))$.

Theorem 5.5. For the infinite, open Bose gas, we have

$$\bar{J}_j^H(q(H)) = \int\frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{2}\right)\hat{g}_j\left(\frac{p^2}{2}\right)\left[b_j\left(\frac{p^2}{2}\right) - q\left(\frac{p^2}{2}\right)\right] \tag{5.18a}$$

$$\bar{J}_j^N(q(H)) = \int\frac{d^3p}{(2\pi)^3}\hat{g}_j\left(\frac{p^2}{2}\right)\left[b_j\left(\frac{p^2}{2}\right) - q\left(\frac{p^2}{2}\right)\right] \tag{5.18b}$$

$$\begin{aligned}\bar{\sigma}(q(H)) = \sum_{j=1}^{r}\int\frac{d^3p}{(2\pi)^3}\hat{g}_j\left(\frac{p^2}{2}\right)\left[b_j\left(\frac{p^2}{2}\right) - q\left(\frac{p^2}{2}\right)\right]\\ \times\left\{\log\left[1 + q\left(\frac{p^2}{2}\right)\right] - \log q\left(\frac{p^2}{2}\right) - \beta_j\left(\frac{p^2}{2} - \mu_j\right)\right]\end{aligned} \tag{5.19}$$

The heat and matter flow densities in the stationary state satisfy the Onsager relations. The entropy production density is positive and convex, and the principle of minimal entropy production in the stationary state holds in the linear approximation.

Proof (sketch). All expressions $|\Lambda|^{-1}J_j^n(q(H_\Lambda))$ $(j = 1,\ldots, r;\ n = H, N)$ and $|\Lambda|^{-1}\sigma(q(H_\Lambda))$ are of the form $|\Lambda|^{-1}\operatorname{tr} f(-\Delta_\Lambda)$, where f satisfies the hypotheses of Theorem 5.1, or is the difference of functions satisfying the hypotheses of Theorem 5.1, by assumption (F). Then, recalling Eqs. (5.5), (5.7a), (5.7b), and (5.15), we find the expressions (5.18a), (5.18b), and (5.19). The Onsager relations for the heat and matter flow densities (5.18) follow from the same argument as in the finite-volume case. Positivity and convexity of (5.19) are obvious. Its minimization by a standard variational method leads to condition (5.16) for all $\epsilon \geqslant 0$. Then proceed as in Theorem 5.4.

Remark. Suppose that the functions $q^t(\epsilon)$ and $q_j^t(\epsilon)$ are eventually decreasing for t in some neighborhood of zero (this would be trivially true if all the functions $\hat{g}_j$ were of compact support). Then we might obtain (5.18) by first taking the infinite-volume limit of $|\Lambda|^{-1}\omega_{q_j^t(H_\Lambda)}(\tilde{H}_\Lambda)$ (same for N_Λ) using Theorem 5.1, and then differentiating the resulting expression under the integral sign (the estimates needed for the justification of this are the same as in Lemma 5.2). Similarly, the limit as $\Lambda \nearrow \infty$ of $|\Lambda|^{-1}S(q^t(H_\Lambda))$ would be[17]

$$\int \frac{d^3p}{(2\pi)^3}\left\{\left[1 + q^t\left(\frac{p^2}{2}\right)\right]\log\left[1 + q^t\left(\frac{p^2}{2}\right)\right] - q^t\left(\frac{p^2}{2}\right)\log q^t\left(\frac{p^2}{2}\right)\right\} \tag{5.20}$$

This is also the entropy density for the infinite-volume state, as defined and computed in Ref. 18. Again, (5.20) can be differentiated with respect to t under the integral sign, with the same estimates as in Theorem 5.4. When the entropy flow densities are subtracted, we obtain (5.19) again.

6. FERMIONS

Here we sketch how the foregoing discussion can be adapted to treat fermion systems, described by the CAR algebra, by using the results of Refs. 20, 21, and 4.

A representation of the canonical anticommutation relations (CAR) over a (complex) Hilbert space $\mathscr{H}$ is a conjugate-linear map $h \mapsto a(h)$ of $\mathscr{H}$ into the bounded operators on a Hilbert space $\mathscr{H}_A$ satisfying

$$a(h)a(h') + a(h')a(h) = 0 \tag{6.1a}$$

$$a(h)a(h')^* + a(h')^*a(h) = (h, h')\mathbf{1} \tag{6.1b}$$

for all h, h' in $\mathscr{H}$. We use the notation $a(h)^\#$ for $a(h)$ or $a(h)^*$, when no distinction is needed. The CAR algebra over $\mathscr{H}$, denoted by $A(\mathscr{H})$, is the (unique up to isomorphism) C^*-algebra generated by $a(h)^\#$, $h \in \mathscr{H}$. The map $h \mapsto a(h)$ is automatically continuous, so we need not introduce "continuous states" as a particular class of states. If B is a positive contraction on $\mathscr{H}$, we denote by ω_B the state on $A(\mathscr{H})$ given by

$$\omega_B(a(k_n)^* \cdots a(k_1)^*a(h_1) \cdots a(h_m)) = \delta_{m,n}\det\{(h_i, Bk_j)\} \tag{6.2}$$

Such states are the quasi-free gauge-invariant states. For any unitary operator U on $\mathscr{H}$, the map α_U on $A(\mathscr{H})$ defined by

$$\alpha_U a(h)^\# = a(Uh)^\# \tag{6.3}$$

extends to a *-automorphism of $A(\mathscr{H})$. Corresponding to each state ω_B and each contraction T on $\mathscr{H}$ there is a completely positive, identity-preserving map $\Phi_{B,T}$ on $A(\mathscr{H})$ satisfying

$$\begin{aligned}\Phi_{B,T}&[a(h_1)^\# \cdots a(h_n)^\#] \\ &= \sum_p \chi(p) a(Th_{i_1})^\# \cdots a(Th_{i_m})^\# \omega_B(a(Dh_{i_{m+1}})^\# \cdots a(Dh_{i_n})^\#)\end{aligned} \tag{6.4}$$

where $D = (1 - T^*T)^{1/2}$, and where the summation is taken over all partitions p of $\{1,..., n\}$ into two sets $\{i_1,..., i_m\}$ and $\{i_{m+1},..., i_n\}$ with $i_1 < \cdots < i_m$ and $i_{m+1} < \cdots < i_n$, and $\chi(p)$ is the parity of the permutation $\{1,..., n\} \mapsto \{i_1,..., i_n\}$. When T is a projection, then (6.4) is a conditional expectation. Due to the structure (6.2) of the states ω_B, a map satisfying (6.4) is determined by its action on monomials $a(h)^*a(h')$. If $T_t = \exp(Kt)$ is a contraction semigroup commuting with B, then Φ_{B,T_t} is a completely positive, strongly continuous semigroup satisfying

$$\Phi_{B,T_t}[a(h)^*a(h')] = a(T_t h)^* a(T_t h') + \int_0^t ds\, (T_s h', BGT_s h) \tag{6.5}$$

where $G = -K - K^*$. More generally, if C is a positive bounded operator on $\mathscr{H}$, with $C \leqslant G$, then there is a completely positive, strongly continuous semigroup Φ_t on $A(\mathscr{H})$, satisfying[4]

$$\Phi_t[a(h)^*a(h')] = a(T_t h)^* a(T_t h') + \int_0^t ds\, (T_s h', CT_s h) \tag{6.6}$$

If B_j, $j = 1,..., r$, are positive contractions, $T_t^j = \exp(K_j t)$ are contraction semigroups with bounded generators, $[B_j, K_j] = 0$ for all j, then Φ_t defined by (6.6), with $T_t = \exp(\sum_{j=1}^r K_j t)$, $C = \sum_{j=1}^r B_j G_j$, satisfies

$$\begin{aligned}&\frac{d}{dt}\Phi_t[a(h_1)^\# \cdots a(h_n)^\#]|_{t=0} \\ &\qquad = \sum_{j=1}^r \frac{d}{dt}\Phi_{B_j T_t^j}(a(h_1)^\# \cdots a(h_n)^\#)|_{t=0}\end{aligned} \tag{6.7}$$

Dynamical semigroups of the form (6.6) have a stationary state ω_{B_∞}, where $B_\infty = \int_0^\infty ds\, T_s^* C T_s$ ($B_\infty = B$ for Φ_{B,T_t}) and all states approach B_∞ under the action of Φ_t as $t \to \infty$ if and only if T_t converges strongly to zero as $t \to \infty$.

The maps on the CAR algebra introduced here are the exact counterpart of the maps on the CCR algebra introduced in Section 2. It is straightforward to adapt to them the proofs of Sections 3 and 4. Remark, however, that in the CAR case one can prove convergence in the norm of the CAR algebra, and that $A(\overline{\bigcup_\Lambda \mathscr{H}_\Lambda}) = \overline{\bigcup_\Lambda A(\mathscr{H}_\Lambda)}$. Also, the thermodynamic properties of the open Fermi gas coupled to several reservoirs can be discussed and proved exactly as in Section 5, with the replacement of (4.6) by

$$b_j(\epsilon) = \{\exp[\beta_j(\epsilon - \mu_j)] + 1\}^{-1}, \qquad B_j > 0, \quad \mu_j \in \mathbb{R} \tag{6.8}$$

and of (5.4c) by[22]

$$S(\omega_Q) = \operatorname{tr}[(Q - 1)\log(1 - Q) - Q \log Q] \tag{6.9}$$

Therefore, it would be pointless to repeat the arguments here. Instead, we take a simple model system (a one-dimensional lattice system with nearest neighbor interactions) and briefly study the situation that occurs when the simplifying assumption (E) on the coupling is dispensed with and different sites are coupled to reservoirs at different temperatures. Let Λ be a finite subset of $\mathbb{N}$, $\Lambda = \{1,\ldots, 2N\}$, $|\Lambda| = 2N$. Let $\mathscr{H}_\Lambda = \mathbb{C}^{|\Lambda|}$ spanned by the vectors

$$\delta_n, \quad n \in \Lambda; \qquad \delta_n(m) = \delta_{nm}$$

and let the one-particle Hamiltonian be

$$H_\Lambda = \sum_{n\in\Lambda} [\delta_n \otimes \bar{\delta}_n - \tfrac{1}{2}\delta_n \otimes \bar{\delta}_{n+1} - \tfrac{1}{2}\bar{\delta}_{n+1} \otimes \delta_n] \tag{6.10}$$

with periodic boundary conditions. The eigenvalues of H_Λ are

$$\epsilon_k{}^\Lambda = 1 - \cos \vartheta_k{}^\Lambda, \qquad \vartheta_k{}^\Lambda = 2\pi k/|\Lambda|, \qquad k = 0,\ldots, |\Lambda|/2 \tag{6.11}$$

and the corresponding normalized eigenvectors are

$$\begin{array}{lll} c_0{}^\Lambda & & [c_0{}^\Lambda(m) = |\Lambda|^{-1/2} \quad \text{for all } m] \\ c_k{}^\Lambda, s_k{}^\Lambda & & [c_k{}^\Lambda(m) = (2/|\Lambda|)^{1/2} \cos(\vartheta_k{}^\Lambda m) \\ & & s_k{}^\Lambda(m) = (2/|\Lambda|)^{1/2} \sin(\vartheta_k{}^\Lambda m) \\ & & k = 1,\ldots, |\Lambda|/2 - 1] \\ c^\Lambda_{|\Lambda|/2} & & [c^\Lambda_{|\Lambda|/2}(m) = |\Lambda|^{-1/2}(-1)^m] \end{array} \tag{6.12}$$

In the thermodynamic limit, $\mathscr{H} = \overline{\bigcup_\Lambda \mathscr{H}_\Lambda} = l^2$, and H_Λ converges strongly to a bounded operator H on l^2 given by

$$(Hf)(n) = \sum_m \int_0^{2\pi} \frac{d\vartheta}{2\pi}\, e^{i\vartheta(n-m)}(1 - \cos\vartheta) f(m)$$

This is essentially the fermion version of the X–Y model.[23,24] The reservoirs can be taken to be infinitely extended lattice systems of the same type or other quasi-free Fermi systems if desired. Let each site be coupled to its own reservoir with a coupling

$$V_\Lambda = \sum_{n\in\Lambda} f_n \otimes \bar{\delta}_n \tag{6.13}$$

where

$$(f_m, U_t^0 f_n) = \delta_{mn} g_n(t) \in L^1(\mathbb{R})$$

Let also the reference state of each reservoir be determined by a function $b_n(H)$, with

$$b_n(\epsilon) = (e^{\beta_n \epsilon} + 1)^{-1} \tag{6.14}$$

The resulting structure for K_Λ is very complicated. It becomes simple when only the sites $n = |\Lambda|/2$ and $n = |\Lambda|$ are coupled to a reservoir. Indeed, for such sites δ_n is orthogonal to all functions s_k^Λ. Let us refer to the site $|\Lambda|/2$ (respectively $|\Lambda|$) as the "left" (respectively "right") end of the chain (due to the periodic boundary conditions, the sites 1 and $|\Lambda|$ are nearest neighbors) and label by an index l (respectively r) the quantities referring to it. With the use of Lemma 3.1 we find

$$K_\Lambda = -|\Lambda|^{-1} \sum_{k=0}^{|\Lambda|/2} \int_0^\infty dt\, [\exp(-i\epsilon_k^\Lambda t)][g_l(t) + g_r(t)] \\ \times (2 - \delta_{k,0} - \delta_{k,|\Lambda|/2}) c_k^\Lambda \otimes \bar{c}_k^\Lambda \tag{6.15}$$

Now K_Λ commutes with H_Λ, but it is not a function of it. It annihilates the linear span of $\{s_k^\Lambda\}$, so that the stationary state is not uniquely determined: for any operator B_Λ of the form

$$B_\Lambda = B_\Lambda^s + \sum_{k=0}^{|\Lambda|/2} \frac{\hat{g}_l(\epsilon_k^\Lambda) b_l(\epsilon_k^\Lambda) + \hat{g}_r(\epsilon_k^\Lambda) b_r(\epsilon_k^\Lambda)}{\hat{g}_l(\epsilon_k^\Lambda) + \hat{g}_r(\epsilon_k^\Lambda)} c_k^\Lambda \otimes \bar{c}_k^\Lambda \tag{6.16}$$

where B_Λ^s is an arbitrary, positive contraction on $\text{lin}\{s_k^\Lambda\}$, the state ω_{B_Λ} is a stationary state for the dynamical semigroup Φ_t^Λ, and Φ_t^Λ can be written as $\Phi_{B_\Lambda, T \cdot t}{}^\Lambda$ In particular, we can choose B_Λ as $b(H_\Lambda)$, where

$$b(\epsilon) = \frac{\hat{g}_l(\epsilon) b_l(\epsilon) + \hat{g}_r(\epsilon) b_r(\epsilon)}{\hat{g}_l(\epsilon) + \hat{g}_r(\epsilon)} \tag{6.17}$$

Hence, the local structure of the coupling does not lead to a local structure of the stationary state. The heat flow coming into the system in the stationary state from the left reservoir is given by [compare (5.7a)]

$$J_l(B_\Lambda) = |\Lambda|^{-1} \sum_{k=0}^{|\Lambda|/2} (2 - \delta_{k,0} - \delta_{k,|\Lambda|/2}) \epsilon_k^\Lambda \\ \times \frac{\hat{g}_l(\epsilon_k^\Lambda) \hat{g}_r(\epsilon_k^\Lambda)}{\hat{g}_l(\epsilon_k^\Lambda) + \hat{g}_r(\epsilon_k^\Lambda)} [b_l(\epsilon_k^\Lambda) - b_r(\epsilon_k^\Lambda)] \tag{6.18}$$

and the heat flow from the right reservoir is

$$J_r(B_\Lambda) = -J_l(B_\Lambda) \tag{6.18'}$$

So we can regard $J_l(B_\Lambda)$ as the steady heat flow through the chain. Note that the result is not affected by the arbitrariness (6.16) in B_Λ. The Onsager relations for the heat flows $J_l(B_\Lambda)$ and $J_r(B_\Lambda)$ are manifestly true. The entropy production is given by

$$\sigma(q(H_\Lambda)) = \sum_{j=l,r} |\Lambda|^{-1} \sum_{k=0}^{|\Lambda|/2} (2 - \delta_{k,0} - \delta_{k,|\Lambda|/2}) \hat{g}_j(\epsilon_k{}^\Lambda) \times [b_j(\epsilon_k{}^\Lambda) - q(\epsilon_k{}^\Lambda)]\{\log[1 - q(\epsilon_k{}^\Lambda)] - \log q(\epsilon_k{}^\Lambda) - \beta_j \epsilon_k{}^\Lambda\} \tag{6.19}$$

Its properties follow from Ref. 1, since the system has a finite number of levels before the thermodynamic limit.

Theorem 6.1. The stationary state $\omega_{b(H_\Lambda)}$, the steady heat flows (6.18) and (6.18′), and the entropy production (6.19) converge in the limit $\Lambda \nearrow \infty$ to

$$\omega_B = \omega_{b(H)}$$

$$J_l(B) = \int_0^\pi \frac{d\vartheta}{\pi} \epsilon(\vartheta) \frac{\hat{g}_l(\epsilon(\vartheta))\hat{g}_r(\epsilon(\vartheta))}{\hat{g}_l(\epsilon(\vartheta)) + \hat{g}_r(\epsilon(\vartheta))} [b_l(\epsilon(\vartheta)) - b_r(\epsilon(\vartheta))] \tag{6.20}$$

$$J_r(B) = -J_l(B) \tag{6.20′}$$

$$\sigma(q(H)) = \sum_{j=l,r} \int_0^\pi \frac{d\vartheta}{\pi} \hat{g}_j(\epsilon(\vartheta))[b_j(\epsilon(\vartheta)) \to q(\epsilon(\vartheta))] \times \{\log[1 - q(\epsilon(\vartheta))] - \log q(\epsilon(\vartheta)) - \beta_j \epsilon(\vartheta)\} \tag{6.21}$$

respectively, where

$$\epsilon(\vartheta) = 1 - \cos\vartheta \tag{6.22}$$

for all continuous functions q with $0 < q(\epsilon) < 1$ $\forall \epsilon \in [0, 2]$. On the other hand,

$$\lim_{\Lambda \nearrow \infty} \Phi_{B_\Lambda, T_t{}^\Lambda} = \mathbf{1} \tag{6.23}$$

and

$$\lim_{\Lambda \nearrow \infty} \omega_{q(H_\Lambda)} \circ \Phi_{B_\Lambda, T^\Lambda_{|\Lambda|t}} = \omega_{q(H)} \circ \Phi_{B,T_t} \tag{6.24}$$

for all continuous functions q with $0 \leqslant q \leqslant 1$, where

$$T_t = \exp(Kt), \qquad K = -\int_0^\infty dt\, [g_l(t) + g_r(t)] \exp(-iHt)\, P^c$$

P^c denoting the projection onto the functions in l^2 whose Fourier transform in $L^2(0, 2\pi)$ is even under the interchange $\vartheta \to 2\pi - \vartheta$. The state $\omega_{q_m(H)}$ of minimal entropy production is characterized by the condition

$$\sum_{j=l,r} \hat{g}_j(\epsilon)\left[\log\frac{1 - q_m(\epsilon)}{q_m(\epsilon)} - \beta_j\epsilon + \frac{b_j(\epsilon) - q_m(\epsilon)}{q_m(\epsilon)[1 - q_m(\epsilon)]}\right] = 0$$

and coincides with the stationary state $\omega_{b(H)}$ to first order in the temperature difference $\beta_l^{-1} - \beta_r^{-1}$.

Proof. Since H_Λ converges strongly to H, and all H_Λ and H are bounded, also $b(H_\Lambda)$ converges strongly to $b(H)$. Thus $\omega_{b(H_\Lambda)}(a(h_1)^\# \cdots a(h_n)^\#)$ converges to $\omega_{b(H)}(a(h_1)^\# \cdots a(h_n)^\#)$ for all $h_1,\ldots,h_n$. The expressions (6.20) and (6.21) are the limits of (6.18) and (6.19), respectively, since $2|\Lambda|^{-1}\sum_{k=0}^{|\Lambda|/2} f(1 - \cos \vartheta_k{}^\Lambda)$ is a Riemann sum for

$$\int_0^\pi \frac{d\vartheta}{\pi} f(1 - \cos \vartheta)$$

for all continuous functions f, and the two terms in (6.18) and (6.19) corresponding to $k = 0$, $|\Lambda|/2$ do not contribute in the limit $\Lambda \nearrow \infty$. We show that $\|K_\Lambda\|$ tends to zero as $\Lambda \nearrow \infty$. Let

$$M = \sup_\epsilon \left| \int_0^\infty dt\, e^{-i\epsilon t}[g_l(t) + g_r(t)] \right| < \infty$$

$$\tilde{K}_\Lambda = -2|\Lambda|^{-1} \int_0^\infty dt\, [g_l(t) + g_r(t)] \exp(-iH_\Lambda t)$$

and let $P_\Lambda{}^c$ denote the projection of $\mathscr{H}_\Lambda$ onto $\text{lin}\{c_k{}^\Lambda\}$. Then

$$\begin{aligned} \|K_\Lambda\| &\leqslant \|P_\Lambda{}^c \tilde{K}_\Lambda P_\Lambda{}^c\| + \|K_\Lambda - P_\Lambda{}^c \tilde{K}_\Lambda P_\Lambda{}^c\| \\ &\leqslant \|\tilde{K}_\Lambda\| + |\Lambda|^{-1} \Big\| \sum_{k=0,|\Lambda|/2} \int_0^\infty dt\, [\exp(-i\epsilon_k{}^\Lambda t)][g_l(t) + g_r(t)] \\ &\quad \times (c_k{}^\Lambda \otimes \bar{c}_k{}^\Lambda) \Big\| \\ &\leqslant 3M|\Lambda|^{-1} \end{aligned}$$

To prove the second part of the theorem, it suffices to show that

$$\omega\text{-}\lim_{\Lambda \nearrow \infty} f(H_\Lambda) P_\Lambda{}^c = f(H) P^c \tag{6.25}$$

for all continuous, bounded functions f. It is enough to consider the matrix elements of (6.25) between vectors δ_m and δ_n for all m, n. We have

$$\begin{aligned} (\delta_m, f(H_\Lambda) P_\Lambda{}^c \delta_n) &= \sum_{k=0}^{|\Lambda|/2} (\delta_m, c_k{}^\Lambda)(c_k{}^\Lambda, \delta_n) f(\epsilon_k{}^\Lambda) \\ &= |\Lambda|^{-1} \sum_{k=0}^{|\Lambda|/2} (2 - \delta_{k,0} - \delta_{k,|\Lambda|/2}) \cos(\vartheta_k{}^\Lambda m) \cos(\vartheta_k{}^\Lambda n) f(\epsilon_k{}^\Lambda) \\ &\xrightarrow[\Lambda \nearrow \infty]{} \int_0^\pi \frac{d\vartheta}{\pi} \cos(\vartheta m) \cos(\vartheta n) f(\epsilon(\vartheta)) \\ &= (\delta_m, f(H) P^c \delta_n) \end{aligned}$$

(the two terms corresponding to $k = 0$, $|\Lambda|/2$ are unimportant in the limit). The properties of entropy production are shown as in Theorems 5.4 and 5.5.

Comments. In this model, the heat flows and the entropy production remain finite in the thermodynamic limit, since the coupling V_Λ to the reservoirs acts only at two sites. Since the temperature gradient $2|\Lambda|^{-1}(\beta_l^{-1} - \beta_r^{-1})$ vanishes as $\Lambda \nearrow \infty$, the system is a heat superconductor in the sense of Ref. 25. The reason is that there is no friction mechanism inside the chain, as in the harmonic crystals considered in Ref. 25. On the other hand, the effect that the coupling to the reservoirs has on the time evolution of local observables does vanish in the thermodynamic limit [compare (6.23)]: indeed, the effect of V_Λ is averaged all over the chain by the free evolution, so that K_Λ is of order $2|\Lambda|^{-1}$. In order to see the effects of the coupling on local observables, one must wait for increasingly longer time intervals as the size of the chain increases [compare (6.24)]. Finally, we remark that the above discussion can be repeated with no essential modifications for any quasi-free Bose or Fermi lattice system with translationally invariant, finite-range interaction.

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